Creative Destruction with Fixed Factor-Augmenting Technical Change: Lego World

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Creative Destruction with Fixed Factor-Augmenting Technical Change: Lego World*

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DRAFT: Comments and suggestions will be deeply welcome.

Abstract

By resolving the knife-edge condition in Solow's neoclassical growth model as interpreted by Uzawa, in which a production technology with a non-unitary substitution elasticity is unable to accommodate capital-augmenting technical change, this paper offers a transformative approach to the analysis of economic growth and technical change. With nature fixing the exogenous supplies of both capital and labor, consistent with the Law of the Conservation of Matter, investment in both physical and human capital becomes constrained to equal quantities of creation and destruction; that is, Hicks invention-neutral technical change creates new vintages that replace obsolete vintages of physical matter. Contrasted with the Solow model in which the steady-state rate of growth is unaffected by the savings rate, in this model the exogenous savings rate drives every dimension of the Schumpeterian creative-destruction process – rates of innovation, depreciation, and investment. Rising living standards result entirely from technology deepening; capital deepening becomes a misnomer. The model is a literal implementation of Shumpeterian growth in which creation and destruction are precisely balanced in the steady state, driven by the warranted rates of savings, technology development, and investment. The steady state is that in which savings, technical change, depreciation, and investment are aligned, so that creation and destruction are precisely balanced consistent with nature's fixed endowment.

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1. Introduction

A primary motivation for Solow's formulation of his iconic neoclassical growth model was to resolve the knife-edge condition of the Harrod-Domar model for which the fixed proportions production technology created the prospect that vast and growing supplies of either unemployed labor or capital would emerge and persist as a by-product of long-run growth. Ironically, in solving the Harrod-Domar knife-edge problem, Solow created another knife-edge problem. As a central feature of a model of "balanced growth," the Solow model's knife-edge condition is conceivably yet more serious than that of unemployment in the Harrod-Domar setting.

A well-known implication of the conventional Solow-Uzawa interpretation of the neoclassical growth model is that once the economy's production technology veers from a unitary elasticity of substitution, i.e., $\sigma = 1$, the steady state is unable to coexist with capital augmenting technical change. As a result of this knife-edged condition, when $\sigma \neq 1$, coincident with the technical capabilities of labor growing exponentially, the absence of improvements in the physical capital stock relegates humankind to a Stone Age populated by super humans.

From the perspective of this paper, this troubling implication of the "workhorse model of macroeconomics" (Acemoglu, 2009, p. 26) results from its failure to broadly incorporate the role of capital in long-run growth. As such, the Solow model (1956) is capable of degenerating to an "AL" model in which the growth of output per capita is simply $g_Y - n = g_y = g_A$, i.e., living standards are driven exclusively by g_A, the rate of growth of purely labor-augmenting Harrod-neutral technical change.[1] With the growth of living standards virtually nothing more or less than the growth of labor efficiency enabled by a perfectly elastic supply of Stone-Age resources, the neoclassical model effectively obscures the real-world role of capital; in particular the roles

[1] We contrast this "AL model" with the well-known AK model in which constant returns to capital, reflecting the condition of constant returns to investment with capital's output elasticity, α, = 1. For the referenced AL model, the conventional steady state $g_Y = g_A$ results for any value of $\alpha < 1$.

of capital-specific technical change and technology deepening, as they operate on and through the process of economic growth.[2]

As shown by Uzawa (1961), only when $\sigma = 1$, i.e., the production technology is Cobb-Douglas, does the Solow-Uzawa model provide the conditions, so that either capital or labor-augmenting technical change – or both – are able to sustain a steady state. For circumstances in which $\sigma \neq 1$, the presence of capital-augmenting technical change causes the income share of either capital (for $\sigma<1$) or labor (for $\sigma>1$) to degenerate to zero, thereby violating a basic tenet of steady-state growth. This paper seeks to resolve this inconsistency between capital-augmenting technical change and balanced growth in the general case, irrespective of the economy's capital-labor substitution elasticity.

To address the apparent impossibility of admitting both capital and labor-augmenting technical change with a non-unitary substitution elasticity, we reformulate the Solow model so that capital and labor are entirely symmetric. With this arrangement, the treatment of physical capital and that of physical labor are identical; that is, given non-zero population growth, the physical supplies of both are exogenously fixed. By acknowledging the fixed size of the physical world, so that investment exactly offsets depreciation, the only avenue for the balanced growth of the effective supplies of capital – and labor – is exogenous Hicks-neutral factor-augmenting technical change.

We demonstrate the set up for this approach both mathematically and through the lens of an isoquant scheme. This approach also introduces the process of technology deepening as both a disembodied phenomenon, as in the case of the Solow model and as a process of embodiment through which new vintages of physical and human capital replace obsolesced vintages. Within this model, technical change becomes inseparable from the rates of savings, investment, and depreciation, including the cost of innovation consistent with a Hicks-neutral steady state. With exogenously-fixed factor inputs, living standards rise exclusively from technology deepening. Capital deepening with investment that alters the physical K-L ratio no longer exists. The result is an extreme form of Schumpeterian creative destruction involving a 1-for-1 substitution of new vintages of capital for obsolete vintages. The steady state in this model is achieved when rates of savings, innovation, depreciation, and investment are precisely balanced.

[2] Note that Solow's assumption of a perfectly elastic supply of capital is analogous to the Harrod-Domar assumption of a perfectly elastic supply of labor.

With a non-unitary substitution elasticity, the introduction of non-zero population growth that alters the distribution of factor augmentation between the human form and its physical environment potentially alters factor-income shares. The regularity of long-run real-world deviations $\geq \varepsilon$ from fixed factor-income shares and a unitary substitution elasticity, may well render the conventional Solow-Uzawa steady state to be an unrealistic and theoretically untenable requirement for a model of long-run growth. By voiding the possibility of factor augmenting technical change for all but physical labor under the special condition of a unitary substitution elasticity, the Solow model impairs our understanding of essential features of the long-run growth process.

Contrary to the Solow model, the rate of savings in this model plays a central role in determining long-run growth. By framing and integrating exogenous rates of savings for physical and human capital with endogenous innovation, depreciation, and investment outcomes, this model implies and requires a certain long-run consistency between savings rates and key dimensions of long-run growth. By assuming fixed supplies of physical capital and labor, as determined by nature in all periods, rates of technology-induced investment and depreciation for both human and physical capital must be balanced. As a literal interpretation of Schumpeterian growth, in the steady state, creativity and destruction – i.e., technical change, savings, investment, and depreciation – must be in perfect balance.

The following section summarizes key efforts in the growth literature to interpret limits on and/or to expand the Solow model to incorporate capital-augmenting technical change. Section 3 reviews the structure of the Solow model under the dissimilar assumptions of Harrod labor-augmenting technical change and Hicks-neutral technical change. Section 4 introduces an isoquant scheme that offers a helpful visual account that distinguishes between Stage I and Stage II of the growth process. Section 5 demonstrates the functioning and limitations of the Solow model within a CES system. Section 6 sets forth the basic assumptions and structure of the creative-destruction model, demonstrating how it functions with Hicks neutrality regardless of the economy's substitution elasticity. Section 7 addresses the issue of population growth, explores the implausible condition of the Solow steady state, and proposes an alternative understanding of balanced growth. Section 8 defines and clarifies the determining role of the savings rate in the model. Section 9 constructs a simple, heuristic model of endogenous growth consistent with the architecture of the model. Section 10 examines various micro-foundations that beg for clarification. Finally, Section 11 sets forth conclusions and reflections, highlighting various contrasts and parallels with the Solow model.

2. Literature Review

In their attempt to make the seminal argument of Uzawa (1961) more transparent, Jones and Scrimgeour (2005) clarify the challenge of incorporating capital-augmenting technical change in the Solow model. Because, with the exception of an exogenous rate of depreciation, the growth of the capital stock in the Solow-Uzawa growth model is entirely endogenous, as explained by Jones and Scrimgeour, capital "inherits" the exogenous growth of labor augmenting technical change. Given the restriction that in the steady state, the effective supplies of capital and labor must grow at the same rates, once the capital side of the ledger has fully inherited labor's augmenting technical change, with $\sigma \neq 1$, space for capital-augmenting technical change to contribute independently to the growth of the capital stock disappears.

We reference Jones and Scrimgeour's (2005) condensed, transparent representation of the Uzawa Theorem in which Uzawa (1961) shows that in all situations but that of a unitary elasticity substitution, a steady state requires technical change to be of the purely labor-augmenting variety. Starting with a general-constant returns-to-scale production technology:

$$Y = f(K,L,t) \tag{1a}$$

Jones and Scrimgeour divide through by Y to obtain:

$$1 = f(K/Y,\ L/Y,\ t). \tag{1b}$$

As Jones and Scrimegour explain**,** in order to maintain balance in the steady state, given that K/Y is fixed, when Y grows so that $g_Y > n$, L, the physical supply of labor, must be supplemented by labor-augmenting technical change, g_A. Hence, steady-state growth requires:

$$1 = f(K/Y,\ AL/Y). \tag{1c}$$

With an exogenous rate of growth of the labor supply, i.e., $n + g_A$, balanced growth requires the endogenously-determined capital stock to "inherit" g_A and n thereby ensuring that $dK/K = g_K = n + g_A$, so that in the steady state the per capita growth of capital, $g_k = g_K - n = g_A = g_y$. In this account, g_k is entirely driven by g_A, labor-augmenting technical change. The inclusion of capital-augmenting technical change results in $g_k > g_A$, i.e., unbalanced growth such that the effective supply of capital exceeds the effective supply of labor. We demonstrate using Fig. 2 in Section 5 why labor-augmenting technical change is uniquely equipped to achieve steady-state consistency when $\sigma \neq 1$.

This limitation requiring technical change to be purely labor-augmenting, thus severely restricting the possibility for technical change to augment the quality of capital, raises serious reservations regarding the assumption of the nature of technical change in long-run growth. According to Solow (2000, p. 31,32), "…it is possible to give theoretical reasons why technological progress might be forced to assume the particular form ("called labor-augmenting") required for the existence of a steady state. They are excessively fancy reasons, not altogether believable." Acemogulu (2009, pp. 62) simply characterizes the assumption as "At some level…distressing."

This inability of the Solow-Uzawa model to accommodate capital-augment technical change when the capital-labor substitution elasticity is non-unitary has challenged decades of researchers to attempt novel approaches to address this difficulty. Acemoglu (2003) analyzes an economy in which firms can undertake both labor- and capital-augmenting technological improvements. In the long run, the economy resembles the standard growth model with purely labor-augmenting technical change, with constant factor-income shares. However, the steady state may be beset by exogenous events, such as tax policy or changes in labor-supply or savings that cause deviations from steady-state or profit-maximizing factor-income shares. In the face of such deviations, consistent with Kennedy (1964) and Samuelson (1965), profit-maximizing firms may undertake investments in capital-augmenting technologies. While such aberrant shocks and the ensuing capital-augmenting technology adjustments are unsustainable in the Solow steady state, the Solow model responds in ways that seek transition paths that lead back to the stable long-run steady state.

Grossman, Helpman, Oberfield, and Samson (2017) look at the case in which $\sigma < 1$, so that, pursuant to capital-augmenting technical change, capital deepening depresses capital's factor-income share causing capital's factor share to trend to zero, i.e., $[\alpha/(1\text{-}\alpha)] \rightarrow 0$. To compensate for this imbalance in which the growth of effective labor exceeds that of effective capital, Grossman et al introduce schooling, the demand for which responds to the growth of the capital stock. That is, provided that schooling is more complementary with capital than with raw labor, i.e., $\partial(F_K/F_L)/\partial s > 0$, schooling serves to augment the demand for capital. With a precisely calibrated set of parameters that enable investments in schooling to offset the change in the capital share resulting from capital deepening when $\sigma < 1$, endogenous schooling is able to rebalance the rates of growth of effective labor and capital, consistent with the steady state in the presence of capital-augmenting technical change.

Starting with the neoclassical growth model into which he incorporates adjustment costs of investment, Irmen (2013) extends Uzawa's Theorem (1961) to allow for capital-augmenting technical change in the presence of steady-state adjustment costs. As a result of adjustment costs of investment, units of current output earmarked for savings are consumed in the process of installing capital, so that capital deepening does not fully inherit the entirety of labor-augmenting technical change. As explained by Irmen, by creating a gap between the evolution of gross capital investments and the capital stock, so that the former grows strictly faster than the latter in the steady-state, capital-augmenting technical change can enter to bridge the gap.[3]

While each of these approaches pries open the door for the inclusion of capital-augmenting technical change in long-runbalanced growth, the disadvantage of each is to compromise the straightforward intuition of the original Solow model. Acemoglu's model (2003) operates outside the Solow-Uzawa steady state, thereby avoiding a fundamental alteration of the model that would allow the model itself to incorporate and generate capital-augmenting technical change. Grossman et al (2017) have the advantage of incorporating an essential factor – schooling – but do so under a limited set of restrictions that may themselves be more irregular than the assumption of a non-unitary substitution elasticity, which they are attempting to address. Finally**,** Irmen (2013) remedies an important shortcoming of the Solow model, which is the assumption that investment costs of adjustment are unrelated to the scale of investment as required for the essential Solow assumption of an infinitely elastic supply of capital. While this is an important insight, as with Acemoglu and Grossman et al, Irmen's innovation is more of a

[3]Li (2016) shows that the essential condition for neoclassical model to have a steady-state growth path is that the rates of change in the *marginal efficiency of capital accumulation* (MECA) and the rate of *capital-augmenting technical change* (CATC) sum to zero.

patch on the Solow model which falls short of enabling capital-augmenting technical change to play as prominent a role in the growth process as labor-augmenting technical change. None of these initiatives creates the global space for critically-needed capital-augmenting technical change.

Moreover, a common condition of all three approaches, one which severely compromises the intuitive, parsimonious attraction of the Solow model, is that each requires precise calibration of the newly introduced condition – optimal investment in capital-augmenting technical change, schooling, or adjustment costs – that entails specifying the optimizing behavior of a representative agent. Writing in response to the proliferation of papers that seek to elaborate on long-run growth modeling, Solow (1994, p. 49) writes: "Maybe I reveal myself merely as old-fashioned, but I see no redeeming social value in using this construction (i.e., the intertemporally-optimizing representative agent)." Arguably, it would be preferable to formulate a model that is able to accommodate capital-augmenting technical change in a way that is no less transparent and parsimonious than that achieved by Solow in 1956.[4]

3. Solow – an overview

To set the stage, we assume the tractable Cobb-Douglas production function with a unitary substitution elasticity (i.e., $\sigma = 1$). We derive the equation for the growth of output per capita in the steady state, first assuming purely labor-augmenting Harrod-neutral technical change, then assuming factor-balanced Hicks-neutral technical change.

The Harrod version. Starting with the Harrod version of the Cobb-Douglas production function:

$$Y = K^{\alpha}(AL)^{1-\alpha}, \qquad (2a)$$

with Y = output, K = the capital stock, A = the labor-augmenting shift parameter, and L = population or the labor force, we convert Eq. (2a) to a rate-of-change version:

$$g_Y = \alpha g_K + (1-\alpha)(g_A + g_L). \qquad (2b)$$

[4] The contributions of Acemoglu (1998) and Li and Bental (2016), are later referenced in Section 8.

Setting $g_L = n$, the rate of population growth, subtracting n from both sides to equate $g_y = g_Y – n$ and $g_k = g_K – n$, and setting $K/Y = V$, as assumed by Solow in the steady-state, so that $g_k = g_y + V\hat{}$, where $V\hat{} = dV/V$, we obtain:

$$g_y = g_A + [\alpha/(1-\alpha)]V\hat{} \qquad (2c)$$

In Eq. (2c), V is the standard Solow steady state condition, $K/Y = s/(n + g_A + \delta)$, where δ is the rate of depreciation.

The Hicks version. For the Hicks version, we start with:

$$Y = (BK)^{\alpha}(AL)^{1-\alpha} \qquad (3a)$$

with B and A serving as technology shift parameters for both capital and labor. Converting Eq. (3a) to rate-of-change form:

$$g_Y = \alpha(g_B + g_K) + (1-\alpha)(g_A + g_L). \qquad (3b)$$

Again, converting to the per capita version, we obtain:

$$g_y = \alpha(g_B + g_k) + (1-\alpha)g_A. \qquad (3c)$$

Taking $g_B = g_A = g_H$, adopting the steady-state restriction of a fixed capital-output ratio, $K/Y = V$, as with the Harrod version, and then solving for g_y gives:

$$g_y = [\alpha/(1-\alpha)]g_H + g_H + [\alpha/(1-\alpha)]V\hat{} \qquad (3d)$$

for which we can combine the g_H terms as:

$$g_y = [1/(1-\alpha)]g_H + [\alpha/(1-\alpha)]V\hat{} \qquad (3e)$$

In addition to the vertical shift in the production function equal to g_H, by raising the marginal product of capital by g_H, the ensuing capital deepening moves the economy along its production function, further elevating the growth of the capital stock by α/(1-α. Again, as with the Harrod version, we assume a steady-state in which V^ = 0; we revisit the role of the steady state equation in Section 8.

The key difference between Eqs. (2c) and (3e) is the coefficients associated with the rates of technical change. Jefferson (2017) refers to these as "technology multipliers" in which, as $[1/(1-\alpha)]g_H$, that for Hicks-neutral technical change is an increasing function of the output elasticity of capital, the economy's endogenous factor, whereas for Harrod labor-augmenting technical change, the technology multiplier is unity irrespective of the relevant factor intensities.

4. Solow Through the Lens of an Isoquant

We introduce an isoquant diagram to elucidate the implications of the impacts of various forms of technical change. Using Jones and Scrimgeour's Eq. (1b) above, we construct our isoquant, $Y_0 = 1$, with K/Y and L/Y measured along the axes. Initially, at (A_0, y_0), K = L = Y = A = 1. Hence, the horizontal line at (K/Y)* = 1 represents a locus of points that is consistent with a Solow steady state. Furthermore, the isoquant $Y_{o,\sigma=1}$ is drawn so as to be consistent with a unitary substitution elasticity. We demonstrate the isoquant analysis first using the Harrod assumption; then using the Hicks assumption.

The Harrod version: We initially use a simple numerical example to demonstrate the two stages associated with the Harrod transition from y_0 to y_1. At A_0, we allow for a one-time increase in labor-augmenting technical change; e.g., during the initial one-year period g_A = 100%, represented in the steady state by the increase in income per capita shown at y_1. In order to simulate the two stages of the adjustment process, we set α, capital's factor income share = 1/3. The isoquant $Y_{0,\sigma=1}$ initially shifts from A_0 to A_1, representing two-thirds of the total increase in y, i.e., $(1-\alpha)g_A$ = 66.7%. With σ = 1, the shift in $Y_{0,\sigma=1}$ from A_0 to A_1 leaves unchanged the relative marginal products of capital and labor, as well as the slope of the price line, i.e., $p_1 = p_0$, thus uniformly increasing the marginal products of both capital and labor by 66.7%. This technology-augmenting impact of g_A constitutes Stage I of the growth process; it establishes the pre-condition for Stage II.

At A_1, the increase in income and savings per worker and the increase in capital's marginal product motivate the Stage II capital deepening, equal to αg_A, thereby moving the economy along the section of isoquant spanning $A_1 - y_1$. At y_1, the steady state, the efficiency increase relative to y_0 results in a halving of the active supply of physical labor. By substituting efficient labor for physical labor, the labor-augmenting technical change leaves the effective labor supply unchanged, so that at y_1, K = AL = Y = 1 and Y/L = y_1 = 2, a 100% gain. At y_1, with continuous labor-augmenting technical change, $g_y = g_k = g_A$.

The Hicks version. We now use the isoquant framework to examine the case of Hicks-neutral technical change, which serves to augment the effective supplies of capital and labor equi-proportionately. Unlike purely labor-augmenting technical change, which only shifts the Y_0,σ=1 isoquant $(1\text{-}\alpha)g_A$ to A_1, as shown in Fig. 1, the equivalent increase in g_H leads to a full factor-augmenting increase of $g_H = \alpha g_A + (1\text{-}\alpha)g_A$, represented by the shift of Y_0,σ=1, from A_0 to A_2.

As with the case of purely labor-augmenting technical change, Eq. (3d) can be interpreted to represent two stages. The first is the Stage I technology-deepening impact, i.e., g_H, in which technical change directly augments the effective supplies of capital and labor equi-proportionally. Given that at A_2, the marginal products rise equi-proportionately, the price line at A_2 is parallel to that at A_0., i.e., $p_2 = p_0$. As with labor-augmenting technical change, Stage I creates the circumstances for the Stage II capital-deepening effect, i.e., $[\alpha/(1\text{-}\alpha)]g_H$ for the full impact of Hicks-neutral technical change, during which capital deepening drives the economy along $Y_{0,\sigma=1}$, to the steady state y_2. With σ = 1, capital-deepening elevates the marginal product of labor by an equal amount thereby leaving the ratio of factor-income shares, wL/rK, unchanged. As shown in Eq. (3e), for Hicks-neutral technical change the full technology multiplier is $[1/(1\text{-}\alpha)]g_H$, versus that of just unity for that associated with purely labor-augmenting technical change.

5. Solow and CES

The Uzawa restriction that requires σ = 1 for the inclusion of capital-augmenting technical change is best shown by taking a CES production function:

$$Y = F(K, L) = [\pi(BK)^{(\sigma-1)/\sigma} + (1 - \pi)(AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)}, \qquad (4a)$$

where π and $(1-\pi)$ are normalized measures of the production technology's capital and labor intensity.[5] From Eq. (4a), we derive:

$$F_K/F_L = [\pi/(1-\pi)][(B/A)^{(\sigma-1)/\sigma}(K/L)^{-1/\sigma}] \qquad (4b)$$

and

$$SH_{KL} = \alpha/(1-\alpha) = [\pi/(1-\pi)][(B/A)(K/L)]^{(\sigma-1)/\sigma}. \qquad (4c)$$

Converting Eq. (4c) to a rate-of-change form gives:

$$[\alpha/(1-\alpha)]\text{^} = [(\sigma-1)/\sigma][(g_B - g_A) + g_k], \qquad (4d)$$

Where $g_k = g_K - g_L$. Equation (4d) demonstrates the role of the unitary substitution elasticity in Uzawa's Theorem. If $\sigma = 1$, such that $(\sigma-1)/\sigma = 0$, the CES functional form imposes no restriction on the bias of factor-augmenting technical change. Eq. (4d) also shows that for $\sigma \neq 1$, by inheriting the contribution of g_A, the growth of g_K leaves no space for $g_B > 0$. Consistent with our distinction between Stage I and Stage II of the neoclassical growth process, we disaggregate Eq. (4d):

$$\text{Stage I: } [\alpha/(1-\alpha)]\text{^} = (\sigma-1/\sigma)(g_A - g_B) > 0 \qquad (4e)$$

$$\text{Stage II: } [\alpha/(1-\alpha)]\text{^} = (\sigma-1/\sigma)g_k < 0 \qquad (4f)$$

As shown by Eqs. (4e) and (4f), any deviation from a balancing of Stage I and Stage II results in a change in the capital-labor factor income share, thereby violating a fundamental condition of the steady state. For the case of $\sigma \neq 1$, the persistence of unequal, imbalanced growth of the effective supplies of capital and labor results in either $SH_{KL} \rightarrow 0$, for the case of $\sigma < 1$, or $SH_{KL} \rightarrow \infty$ for $\sigma > 1$. We use Fig. 2 to demonstrate why with $g_B = 0$, all values of $g_A > 0$ result in a sustainable Solow-Uzawa steady state.

[5] See León-Ledesma et al (2009) for a discussion of the purpose of normalizing a CES function.

In Fig. 2, the shift in Y_0,σ<1 to B_1 due to $g_A > 0$ along the ray for which the factor demand K-L is fixed augments the effective supply of labor and the productivity of capital. The increase in the effective supply of labor causes a spike in the rental price of capital, as shown at p_1', thereby causing α/(1-α) to increase to α'/(1-α'). The increase in capital's output elasticity, α, shifts the lower half of the isoquant $Y_{0,\sigma<1}$ back along the K/L ray thereby reducing the contribution of g_A relative to g_k. For Stage I, the contribution of g_A is diminished from $(1-\alpha)g_A$ to $(1-\alpha')g_A$, i.e. from B_0-B_1 to B_0-B_1' as σ transforms from 1 to < 1.[6]

By increasing α to α' and thereby reducing the Stage I contribution of g_A as shown in Fig. 2, the case of σ < 1 compensates by increasing the contribution of capital deepening from αg_A to $\alpha' g_A$, thereby swinging the upper half of the $Y_{0,\sigma<1}$ isoquant left, until the capital deepening causes it to become tangent with $Y_{0,\sigma=1}$ at y_1. At y_1, capital has fully "inherited" g_A, thus restoring the steady-state K:Y ratio. As shown by Eq. (4f), during Stage II, α/(1-α) decreases, so as to compensate for the previous shrinkage of labor's factor-income share during Stage I as determined by Eq. (4e).

Eqs. (4d) summarizes the Stage I-Stage II adjustments shown in Fig. 2 associated with σ < 1 and factor-augmenting technical change. Conditional on $g_B = 0$, the analysis demonstrates that with $g_k = g_A$ in the steady state, the Stage I and Stage II effects of technical change consistently balance so as to enable the Solow-Uzawa steady state.

We also use Fig. 2 to demonstrate the Hicks-neutral case with σ < 1 in which the isoquant $Y_{0,\sigma<1}$ shifts from B_0 to B_2. With Hicks-neutral technical change leaving the factor-augmenting ratio, A/B, unchanged at B_2 and with the initial K/L ratio also unchanged, the Stage I technology deepening leaves the factor-income shares unchanged. However, at B_2, with the marginal product of capital rising, the Stage II capital deepening commences. Along $B_2 - y_2$', with K-L as net complements, the increase in capital's effective supply results in $dp/p > g_K$, so that [α/(1-α)]^ < 0. In the absence of the initial Stage I having elevated capital's factor income share, as it did with purely labor-augmenting technical change, the Stage II capital-deepening effect of Hicks-neutral change drives the economy to y_2', where reductions in capital's factor-income share persist. In this situation, with on-going Hicks-neutral technical change and $g_k > 0$, α/(1-α) → 0, thus defying a steady-state. This account illustrates the dilemma of Hicks-neutral technical

[6] At K/L, the slope of the Cobb-Douglas isoquant is dK/dL = -[(1-α)/α]K/L; that is, for a fixed K:L ratio, as α increases, the isoquant flattens out.

change specifically and capital-augmenting technical change generally when the economy's substitution elasticity is less than unity.

For the case in which $\sigma > 1$, we simply reverse the inequalities in Eqs. (4e) and (4f). For the Hicks-neutral case, $g_A - g_B = 0$, as with all values of the substitution elasticity, the factor income shares are unchanged by the Stage I isoquant shift. As shown by Stage II, for $\sigma > 1$ the Stage II capital deepening causes capital's factor-income share to grow explosively, thereby violating the conditions of balanced growth. The key purpose of this section is to demonstrate the two distinct stages of the process of economic growth involving factor-augmenting technical change and capital deepening. The Stage I-Stage II technology-deepening versus capital-deepening distinction is essential for understanding the functioning of the creative-destruction model that we introduce in the next section.

6. The fixed physical world.

How can the system accommodate capital- and labor-augmenting Hicks-neutral technical change while also subject to the condition $\sigma \neq 1$? We enable this with a fundamental change in the conception of technical change and capital deepening, founded on two critical assumptions:

> Assumption One: *Consistent with the Law of Conservation of Matter, the total supply or expanse of physical matter, i.e., nature's endowment, whether embedded in the human form or its physical setting is fixed.*

The second assumption is:

> Assumption Two: *With $n = 0$, the sole source of growth of the effective supplies of capital and labor is through technology deepening resulting from Hicks-neutral technical change.*

Together these assumptions imply the following restrictions: first, consistent with Assumption Two, net investment, g_K*, represents the embodiment of Stage I technology deepening, g_H. Second, consistent with Assumption One, depreciation, δ_K, and replacement investment, g_K', represent Stage II capital deepening, resulting in zero net physical capital deepening, i.e., $g_K'' = g_K' - \delta_K = 0$.

Assumption One, strictly speaking a straightforward description of our physical world, is transformative. Rather than viewing our physical world as continuously expandable, let alone expandable with an infinite supply elasticity as assumed in the Solow model, in accord with the Law of Conservation of Matter this condition acknowledges the overall expanse of the physical world as fixed. Hence, physical capital deepening no longer exists; the process of (net) investment it is more suitably characterized as technology deepening. Regardless of whether embedded in the human form or in its physical setting, physical matter is assumed to be deeply malleable, so that the human and physical worlds can be continuously transformed through on-going technological advance.

Consistent with this assumption, the function of the so-called capital-deepening portion of the growth process, i.e., $[\alpha/(1-\alpha)]g_H$ in Eq. (3d) and Fig. 1, is simply to replace the depreciated or obsolesced capital stock with a physically equivalent amount of investment. Given that the totality of physical matter is fixed and indestructible, what does it mean for physical capital to "depreciate" so as to require an exact replacement investment? In our model, technology deepening investment includes two forms: i) *creation investment* consists of the resources used to invent and develop new ideas and technologies, including R&D expenses, and ii) *replacement investment* consists of the resources used to "destroy", retire, and replace the physical capital whose net use value falls below that of new goods and services that come on line as a result of the innovation of g_H, e.g., horse-drawn carriages resulting from the automobile; landlines resulting from the mobile phone.[7] Technology deepening resulting in factor augmentation includes both the upgrading of physical matter that presently actively creates value added, and also such matter whose effective economic contribution has heretofore been zero, such as previously undiscovered energy resources whose economic value is transformed from zero to g_H. Investment, assumed in the aggregate to exactly replace the depreciated, or lower-value physical capital, results in technology deepening that augments annually the economic value of the physical capital by g_H. Technical change sets the stage for the conversion and reconfiguration of the fixed supply of physical matter. Although it is unlikely that Schumpeter conceived of his "creative destruction" metaphor in terms of an exact 1-for-1 balance between creativity and

[7] Hulton and Wykoff (1981) estimate that the largest portion of depreciation results from obsolescence.

destruction, we assume this balance, with further elaboration for human capital as explained later in this section.

The step-by-step argument is set forth below. However, a summary of the argument can be gleaned from the previous section. In Fig. 2, regardless of the value of σ, the Stage I technology-deepening impact drives the economy to B_2, where, as a result of the Hicks-neutral technical change, the marginal products of capital and labor rise equi-proportionately, thus leaving the relative marginal products unchanged at the original K:L ratio. As described above, in the conventional Solow model, with $\sigma \neq 1$, the resulting Stage II physical capital deepening, i.e., $g_K = [\alpha/(1-\alpha)]g_H$, causes the economy to migrate to the horizontal ray $(K/Y)^* = 1$, exclusive of the unattainable steady state at y_2. Our innovation is that by interpreting $g_K' = [\alpha/(1-\alpha)]g_H = \delta_K$ as representing equivalent amounts of both depreciation and replacement investment, net physical investment serves to replace old technologies with new technologies, thereby leaving the physical K:L ratio and balanced growth path unaltered as shown at B_2 with its projection to y_1.

By depreciating or destroying the Stage II process resulting from factor-augmenting technical change, so that net physical investment = 0 and technology-deepening investment = g_H, the effective supply of capital no longer "inherits" g_A, rather it responds only to g_B. Nonetheless, given that $g_A = g_B = g_H$, in the steady state $g_K = g_A$, as if the process of economic growth were driven by purely labor-augmenting technical change consistent with the general Solow-Uzawa steady state. We describe this process of depreciation and technology deepening, including that for labor, more fully below and in Fig. 3.

Setting V^ = 0 in the steady state, using Eqs. (3d) and (3e) and applying the steady-state constraint $g_K = g_Y$, in which we accept Solow's assumption that the supply of physical capital is perfectly elastic, we write g_K as:

$$g_K = g_H + [\alpha/(1-\alpha)]g_H = [1/(1-\alpha)]g_H. \qquad (5a)$$

Again, $1/(1-\alpha)$ represents the technology multiplier for g_H, the measure of gross Hicks-neutral technical change, as it drives the growth of gross investment, g_K, and gross output, g_Y. As with the technology multiplier shown in Eq. (3d), the technology multiplier for gross investment consists of two sources of growth in the effective supply of capital. The first, represented by g_H, is disembodied technical change that elevates the overall productivity and effective supply of the capital stock; the second source, represented by $[\alpha/(1-\alpha)]g_H$, represents embodied technical change that materializes through physical capital deepening.

We represent these changes in Fig. 3, which incorporates several refinements relative to Figs. 1 and 2. These are, first, that the fixed effective capital-output ratio is represented as $(BK/Y)^* = 1$. As explained below in the following sub-section, the counterpart fixed effective labor-output ratio is $(AL/Y)^* = 1$, so that the relevant fixed K-L ratio is now transformed to BK/AL, i.e., the fixed effective factor ratio. Throughout, in Fig. 3, B/Y and A/Y are fixed. In the conventional Solow model, AL/Y and K/Y are fixed. Here, AL/Y and BK/Y are fixed as shown in Fig. 3.

Within this setting, the Stage 1 Hicks-neutral technical advance leaves the relative marginal products and factor-income shares unchanged at C_1. Thereafter, with our assumption of a fixed physical stock of capital, rather than the conventional Stage II capital deepening that transpires along C_1 to y_1 for the case of $\sigma = 1$ and along C_1 to y_1' for the case of $\sigma < 1$, the economy transitions to y_2. At $(BK/Y)^*$, the new technology advance, g_H, has fully transformed the extant physcial capital stock. Given our fundamental assumption that the physical supply of capital is fixed, capital deepening is exactly offset by an equivalent amount of depreciation, $\delta_K = [\alpha/(1-\alpha)]g_H$. With the growth of physical capital, $g_K' = [\alpha/(1-\alpha)]g_H = 0$, the net replacement effect, i.e., the effective net growth of capital, is g_H. In the Solow steady state, the relative physical marginal products of both capital and labor are unchanged.

Labor-augmenting technical change. The above discussion describes a model in which we assume that the endogenous dynamic of capital-augmenting technical change resulting in balanced depreciation and replacement investment applies only to physical capital. In fact, our assumption is that nature imposes fixed supplies on all physical matter, both physical and human, so that changes in either result only from exogenous non-zero population growth. Absent exogenous population growth, we require that, as with the physical world, the augmentation of human capabilities be driven purely by technological change, not additive physical investment. As such, human capital enters our model synonymously with physical capital.

With fixed adult cognitive capabilities, such that the supply per capita of the building blocks for human capital is fixed, the cognitive content of the human brain can be continuously updated.[8] As such, knowledge creation and memory entail rearrangements of existing neurons involving the creation of new synapses; this is the neuro-psychological phenomenon of Hicks-invention-neutral human capital augmentation, i.e., $g_A = g_H$. The entire spectrum of adult learning, ranging from introspection to frontier invention invokes the process of creative destruction in which new vintages of human capital investment replace obsolete or unused realms of understanding.

Once the steady state is fixed for physical capital at y_2, the elements of A, L, and Y are also fixed at y_2 and k_2, i.e., the horizontal projection from C_1. The yet-to-be-resolved issue is how to assess the impact of g_A on the depreciation of human capital. We do this through the following mental experiment. Suppose that the physical human form were endogenous to technical change, i.e., either through population growth as in the Malthusian world or through an enlargement of individual cognitive capabilities, i.e., the brain. This assumption would place labor and capital on the same platform, enabling the supplies of both to be perfectly elastic. Under this assumption, we can calculate how large a percentage of labor's steady state physical growth must be destroyed, depreciated, or replaced consistent with humankind's cognitive constraint. ,

As with physical capital, under the assumption of a perfectly elastic supply of human capability, in order to create the labor-equivalent to Eq. (5a), we set $g_L = g_Y$ and solve Eq. (3b) for g_L:

$$g_L = g_B + [(1-\alpha)/\alpha]g_A = (1/\alpha)g_H \qquad (5b)$$

Symmetric with Eq. (5a) that identifies the degree of physical capital deepening that needs to be replaced in order to accommodate new vintages of capital investment, i.e., $[\alpha/(1-\alpha)]g_B$, Eq. (5b) identifies the requisite portion of human capital deepening that is replaced in the process of human capital technology deepening, in order to sustain balanced growth. That is, *balanced growth is the condition in which the effective supplies of **both** the physical world and human capabilities grow, consistent with the constraint of the fixed expanse of the material world.*

[8] See Herbert Simon's work on bounded rationality.

In Fig. 3, this total multiplier effect of labor-augmenting technical change with a perfectly elastic effective supply, i.e., $(1/\alpha)g_{A,}$ is depicted along the vertical axis at k_1. The conditions of the model - fixed supplies of physical capital and physical-cognitive labor, Hicks-neutral technical change, and offsetting physical quantities of depreciation and investment - establish the equilibrium at C_1. At C_1, with the associated projections to y_2 and k_2, during Stage I, disembodied factor augmentation transpires purely through technology deepening. With BK/AL fixed, such that the ratio of the effective factor supplies is fixed, the conventional Solow-Uzawa steady state is satisfied irrespective of the magnitude of the economy's capital-labor substitution elasticity.

Note that with this model that starts with the Law of Conservation of Matter and an exogenously fixed supply of physical capital, with a non-unitary substitution elasticity, purely labor-augmenting Harrod-neutral technical change is inconsistent with the Solow-Uzawa steady state. The intuition behind this statement may well be the same that has caused disquiet for Solow, Acemoglu, Grossman and others seeking to innovate ways of accommodating capital-augmenting technical change within the framework of the Solow model.

7. Population growth vs. the "Black Hole" version

In the previous sections, we assume that the rate of population growth, n, = 0. Before directly examining the implications of non-zero population growth, causing the proportional distribution of nature's fixed physical matter to redistribute between the human and physical worlds, we examine the "Black Hole" version of the model. In this version, we eliminate the distinction between capital and labor, assuming a homogenous structure of physical matter, represented as Earth (E) and a single exogenous rate of technological advance (T). With non-zero population growth, the phenomenon of creative destruction remains consistent with nature's overall endowment constraint. This results in the most simple of functional forms, i.e.,

$$Y = TE,$$

in which the mass, volume, and numeric content of E includes the entirety of the physical matter embedded in both human and non-human form and T represents the technological advances that emerge both from nature's workshop and that of the human intellect. Given the available estimates of 3.5 x 10^{50} for the number of atoms forming the Earth and with the total value of the Earth, including its population, estimated at \$5.0 x 10^{15},[9] the implied value per atom is \$7.5 x 10^{-34}. Two caveats are in order. The first is that prior to 200,000 years ago when humans first appeared on the surface of the Earth,[10] the assessed value of this collection of atoms was zero. Hence, the intrinsic value of the average atom has risen dramatically in a comparatively short time. The second caveat is the fact that there exists substantial heterogeneity within the atomic structure of the Earth and its inhabitants. For example, the most expensive elements include the following costs per gram: Plutonium (\$4,000), Painite (\$9,000), Taaffeite (\$20,000), Tritium (\$30,000), Diamond (\$55,000), Californium (\$25-27 million) and Antimatter (\$62.5 trillion).[11] The average annual income per gram of weight of the average American adult is 75 cents discounted to approximately \$10 over a 40-year working lifetime.[12]

Notwithstanding these differences, a logical extension of our model allows for it to be condensed to a homogeneous, one-sector account of global progress. However, three conditions recommend that we separate the physical world and human condition into two distinct forms of factors of production. The first is that the distinction provides the home for human capital; as such, it is one of the two essential loci of innovation: the human intelligence workshop, which complements nature's innovation workshop as the driver of global innovation. The second distinguishing feature of the human form is that in a human-centric universe, "man is the measure of all things."[13] As such, human convention dictates that the fruits of technical change, notably that of greater productivity and income, are measured on a per person or per worker basis. Increases in living standards are the sine qua non of a meaningful growth model. Finally, collapsing all matter into a single mass would come at the cost of obscuring the critical roles in the growth process of investment, depreciation, population growth, and, as outlined in the next section, for both physical and human capital savings. Hence, we acknowledge population

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[9] https://www.treehugger.com/natural-sciences/new-formula-values-earth-at-5000000000000000.html

[10] http://www.bbc.co.uk/nature/history_of_the_earth

[11] https://www.google.com/search?q=how+much+is+an+atom+worth%3F&rlz=1C1GGRV_enUS753US753&oq=how+much+is+an+atom+worth%3F&aqs=chrome..69i57j0l2.11290j0j8&sourceid=chrome&ie=UTF-8#q=most+expensive+element+in+the+periodic+table

[12] Using an annual discount rate of 3%.

[13] A statement by the ancient Greek philosopher Protagoras. As long as humans are the sole source of measure and arbitration, humankind is the measure of all things. Broadly defined, humankind's objective function may incorporate intra- and inter-generational altruistic considerations.

growth as a critical and distinct feature of the process of technology deepening and economic growth.

In principle, population growth is constrained by the number of atoms in the world, either directly or indirectly, including those required of the physical environment to sustain and enhance life. From the perspective of the distinction between physical and human capital, each individual represents the transformation of atoms from the physical capital side of the ledger to the human side of the ledger. On Nature's workbench, i.e., the process of evolution, this transformation transpired over millions of years. Given that the approximate number of atoms in the world is 3.5×10^{50}, while the approximate average for those residing in a single person is 7×10^{27}, with a current human population of approximately 7 billion persons, i.e., 7×10^{9}, the world's population accounts for about 5×10^{37} atoms.[14] These magnitudes imply a physical capital-labor ratio of roughly 7×10^{14} atoms per person.

Relative to the aggregate, i.e., a figure of the order of magnitude of 10^{50}, the human population accounts for just 7×10^{-15} of the total physical size of the world. As a result, population change that alters the number of atoms embedded in the totality of humankind, represented in the denominator of K/L, exercises a trivial impact on the numerator. Hence, with population growth, the K-L ratio changes as L_0e^{-nt} – the conventional measure of population growth.[15]

Given that measured in terms of atoms, the ratio between those embedded in the physical and human worlds is of the magnitude 10^{14}, it may seem implausible that the ratio of factor-income shares, $\alpha/(1-\alpha)$, should be so many orders of magnitude smaller. Notwithstanding the examples of high-value physical matter cited above, this disparity reflects the far-greater value-creating capacity of the matter embedded in the average human than an equivalent mass randomly drawn from the physical world.

With $n > 0$, $\sigma \neq 1$, and with Hicks-neutral technical change, the increase in the effective supply of labor causes labor's factor income share to be unstable, i.e., $[\alpha/(1-\alpha)]^\wedge \neq 0$.[16] However, before conceding that our model deviates in some significant practical manner from the

[14] http://education.jlab.org/qa/mathatom_04.html

[15] With the advent of space exploration, it is likely that the effective K:L ratio will be growing over time.

[16] AK models with both physical and human capital generally assume n = 0. However, as explained by Lebre de Freitas, http://sweet.ua.pt/afreitas/growthbook/Part%20I/mlfchap5.pdf , p. 17, it is possible to represent H as H = hL in which h is a quality-adjustment measure of H. As a result, H is a measure of the quality-adjusted labor supply. For n ≠ 0, however, this measure of H is not consistent with the assumption of Hicks-neutral technical change.

conventional Solow steady-state, we share the following two points, both of which draw on the empirical context of the hypothetical steady-state. The first is that the Solow steady state depends on the stability of the steady state equation, i.e., $K/Y = s/(n + g + \delta)$. A change in any one of these parameters voids the steady state as the model searches for its transition to a new steady state. Also, by the time convergence to a new steady state has been achieved,[17] simulated in the literature to span up to a generation, new shocks to s, n, g, and/or δ are certain to transpire. Arguably, given the conditionality of the stability of the four critical steady-state parameters, notwithstanding its laudable convergence properties, the Solow economy never achieves its heralded steady state.

The second empirical issue is based on the research of Pritchett (1996), who uses a variety of data sets spanning substantial samples of OECD and developing economies to investigate the relationship between population growth and key macroeconomic conditions relating to long-run growth. In his study, Pritchett finds no consistent pattern of a relationship between population growth and measures of capital per worker, years of schooling, or TFP, i.e., the per capita measures A/L and K/L appear to be unaffected by population growth. Moreover, to the extent that population does affect macroeconomic outcomes, Pritchett concludes that the determining factor is the labor force structure rather than population growth per se. Based on these results, Pritchett debunks the following "powerful intuition":

> …if there is a fixed amount of stuff (land, capital, savings, water, budget for education, or whatever) then if there are more people to share the stuff, the average stuff per person must go down. In the face of this compelling line of reasoning, the more ambiguous and tenuous theory based on endogenous behavioral responses to population pressures, and complicated econometric work never has a chance at persuasion. However, the evidence presented here suggests the basic premise is wrong: there is not a fixed amount of stuff. (p. 25)

Our interpretation of Pritchett's findings is that although there is a "fixed amount of (physical) stuff," notwithstanding this physical fixity, changes in population, i.e., reallocations of the overall "stuff" between its physical and human forms, do not significantly affect the macroeconomic outcomes relevant to long-run growth.

[17] See "Intermediate Macroeconomics: Economic Growth and the Solow Model," Eric Sims University of Notre Dame Fall 2012.

Perhaps the most instructive way to view the matter of increments and instabilities in nature's physical K-L endowment is to explore again the implications of the disparity between the factor income share, $\alpha/(1-\alpha)$, and the physical K-L ratio derived above of 10^{14}. This disparity indicates that the overwhelming vast portion of the magnitude difference in factor income share is due to the differential accumulations of human and physical capital, i.e. A and B. That is A/L is vastly greater than B/K. The overwhelming share of AL is embedded in human capital, not in unskilled, illiterate physical labor.[18] In his finding that the structure of the labor force, not population growth, explains variations in the relevant macroeconomic variables, Pritchett is most likely identifying the deep structural and generational shifts that alter the stock of a nation's human capital, not the annual count of the additions to a nation's physical population. Notwithstanding the extreme conditionality of the Solow-Uzawa steady state, if indeed, given the divergent values of K and L, a one-time change in population growth persists for multiple generations, new steady state factor income shares will emerge. Otherwise, the Solow-Uzawa "steady state" is always a work in progress.

Given Prichett's finding consistent with the intuition that the contribution of labor to production should be represented by labor force structure rather than the rate of population growth at a moment in time, we write g_L as:

$$g_L = \int_{t-65}^{t-20} (n_t/45) \tag{6b}$$

That is, we cumulate annual rates of the 45-year lifespan of the active workforce.

The key point of this section is that regardless of the role of population as it affects the ratio BK/AL, unlike the Solow model, our model does not rely on the assumption that the economy functions as a steady state mechanism defined by fixed factor income shares. As shown in the following section, the steady state in our model requires consistency of the capital-labor rates of savings with the prevailing rates of technical change, depreciation, and investment.

[18] Moreover if gA = gB = gH an A >> B, the it must be the case that the new born or new labor force entrant is vastly more valuable than the work in progress piece of physical capital, i.e., at some point their respective in-utero experiences leads to a starting gate that normalizes the speed and distance of the technology-deepening race course.

That is, our steady state requires that consistent with nature's fixed endowment, technology deepening, creation, and destruction must be in plausible balance.

8. Savings in the steady state

Explicit implications for physical and economic savings behavior arise from this model. We start with the following capital-augmenting equation of motion:

$$d(BK)_{t+1} = s_K Yt - \delta_K(BK)_t + g_B(BK)_t \quad (8a)$$

In the steady state, dividing through by BK_t yields:

$$d(BK)/(BK) = g_K^* = g_B + g_K'' = s_K[Y/(BK)]_t - \delta_K + g_B, \quad (8b)$$

where gK* = net investment, and g_K'' represents net physical replacement investment, i.e., replacement investment less depreciation, such that $g_K'' = g_K' - \delta_K = 0$. Hence gross investment $g_K = g_B + g_K'$.

Eq. (8b) can be adapted to represent two distinct versions of the steady state: one in which technical change is disembodied; the other in which technical change is embodied in new investment. For the case of disembodied technical change, g_B, representing exogenous technology deepening, survives on the right-hand side of Eq. (8b). The resulting steady state is consistent with the growth paths as represented by A_2, B_2, and C_1 in Figs. 1, 2, and 3 respectively in which the technical change transpires in disembodied form in Stage I. Incorporating these restrictions and substituting $\delta_K = [\alpha/(1-\alpha)]g_B$; then solving for the replacement savings rate, s_K', yields:

$$g_K'' = s_K'[Y/(BK)]_t - \delta_K = s_K'[Y/(BK)]_t - [\alpha.(1-\alpha)]g_B = 0 \quad (8c)$$

so that

$$s_K' = [\alpha/(1-\alpha)]g_B(BK/Y). \quad (8d)$$

In Eq. (8d), s_K' is the rate of savings required to exactly replace the depreciated physical capital so as to establish the steady state at y_2 in Fig. 3. This outcome assumes that new investment conveys none of the new vintages of technology, that is, all of the depreciated capital is replaced with technology-equivalent investment, so that the entirety of g_B is absorbed through disembodied means that transpire through Stage I.

The alternative scenario is that of embodied technical change. Under this scenario, the gross rate of savings, s_K, is required to replace the depreciated stock of physical capital while also financing the purchase of the new vintages whose economic values have risen by g_B. For the purpose of representing the savings rate consistent with full embodiment, g_B drops from the right hand side of Eq. (8a), so that rather than materializing costlessly, technical change transpires through the saving process. Hence with capital augmenting technical change, so that in the steady state g_B transpires through the investment process, we amend Eq. (8b) as:

$$g_B = s_K(Y/BK)_t - \delta_K \qquad (8e)$$

Again setting $\delta_K = [\alpha/(1-\alpha)]g_B$, substituting into Eq. (8e), and solving for s_K yields:

$$s_K = [1/(1-\alpha)]g_B(BK/Y) \qquad (8f)$$

The counterpart net rate of savings is:

$$s_K{}^* = g_B(BK/Y). \qquad (8g)$$

Note that whether technical change is embodied as in Eq. (8g) or disembodied as in Eq. (8d), $g_Y = g_H = g_B$ as shown at y_2 in Fig. 3.

Given that in the steady state $g_Y = g_B = g_H$, for the standard Solow steady state with Hicks-neutral technical change and letting v = (BK/Y) represent the economy's capital intensity, we can rewrite Eq. (8g) as $s_K{}^* = g_Y/v$ or $g_Y = s_K{}^*/v$. While having arrived at this result through very different means, the structure of Eq. (8g) is identical to those of Harrod's specification of long-run growth (1961) and Piketty's "second fundamental law of capitalism" (2015).

Ironically, as shown in Annex II, these results were arrived at through a model that can be interpreted as an amended version of the Solow model.

Drawing on the analysis for human capital in the previous section, the derivation of the warranted rates of human capital savings is similar to those just derived for physical capital. For the Stage I disembodied scenario, the implied human capital savings rate is:

$$s_L' = [(1-\alpha)/\alpha)]g_A(AL/Y) \tag{8h}$$

whereas that for the implied gross rate of savings required to embody the technical change in new vintages of human capital is:

$$s_L = [(1/\alpha)]g_A(AL/Y) \tag{8i}$$

and that for the net rate of human capital savings is:

$$s_L^* = g_A(AL/Y). \tag{8j}$$

We note that regardless of whether technical change exclusively transpires as disembodied change in Stage I or it materializes as embodied in Stage II, the result is the same Solow steady state outcome in which $g_Y = g_B = g_A = g_H$, i.e., capital and labor factor augmentation are balanced resulting in balanced growth.

As Eqs. (8d) and (8g) both show, the more technology- and capital-intensive the economy, i.e., the greater BK/Y, the large the requisite savings rate for physical capital. As an example, we take the historic 3% rate of growth of the U.S. economy as a proxy for g_H and designate the BK/Y ratio as 3 and capital's income share as one-third. Using Eq. (8d), to compute the rates of depreciation and replacement investment yields a rate of 4.5%. For gross savings, Eq. (8f) yields a rate of 13.5%. These results imply a ratio of gross to net savings of two thirds.

Based on Eqs. (8d) and (8g), we are also able to compute the net rate of savings for physical capital as $s_K^* = g_B(BK/Y)$. Similarly, Eqs. (8j) infers a net rate of savings for human capital as $s_L^* = g_A(AL/Y)$. That is, our model indicates that the net rate of savings for physical capital savings, $s_K^* = (1-\alpha)s_K$, whereas the net rate of savings for human capital, $s_L^* = \alpha s_L$.

Eqs. (8i) – (8j) imply rates of savings that are warranted by the rates of depreciation, investment, and technical advance of human capital that may seem surprisingly high. For the U.S., using Eq. (8g) and taking the historic 3% rate of growth as a proxy for capital-augmenting Hicks-neutral technical change, setting AL/Y, an estimate of the total labor-output ratio, equal to 6, and setting human capital's income share at two-thirds implies seemingly improbable savings rates for human capital of 36% and 54%, yielding a gross- to replacement savings ratio two-thirds larger than that for physical capital. The implied net rates of savings for physical and human capital are 9% and 18% respectively.

Against the background of our creative-destruction model, comparisons of the savings rates that are implied by our model and those represented in U.S. national income accounts are neither comparable nor complete. The NIA do not attempt estimates for savings and depreciation for human capital in the U.S. economy. Moreover, to the extent that the NIA includes R&D expenditure in its accounts, these are recorded as savings and investment for physical capital, whereas such expenditures are also substantially labor-augmenting as represented by g_A.

A 2007 OECD report indicates that in the OECD countries, approximately 54% of savings is dedicated to human capital investment.[19] Hence, even using accounting methods that reflect existing accounting conventions, the results implied by our model are consistent with the established empirical estimates that suggest that rates of savings for human capital exceed those for physical capital. The empirical implementation of this model clearly requires a fresh look at national income accounting standards in the U.S. and other countries. Our model, for example, forces consideration that three meals a day and the time spent reading this paper qualify respectively as embodied replacement human capital and technology deepening.

Given that for the U.S. the estimated ratio of AL/BK is 2:1, the fact that the estimated ratio of output elasticities is $\alpha/(1-\alpha)$, i.e., 1:2 implies a virtual equalization of marginal products: $\alpha(Y/BK) \sim (1-\alpha)(Y/AL)$. Hence, gross savings rates in the 4:1 labor-capital ratio are consistent with a steady state that tends to equalize returns to investment in physical and human capital. A

[19] https://www.oecd.org/insights/humancapitalhowwhatyouknowshapesyourlife.htm

key implication of this result is that by omitting technology deepening for both physical and human capital, the Solow model misses the critical dynamic force driving long-run growth.

Whereas in the Solow model, in the absence of capital-augmenting technical change, either physical or human, technical change is necessarily disembodied, our model enables a clear distinction between embodied and disembodied technical change, as well as the requisite savings rates associated with each. In this analysis in which physical and human technology deepening are the sole vehicles for long-run economic growth, we assume that technology must be embodied in labor, capital, or both in order to affect the economy. Blueprints, patents, and journal articles may reside on line or in the archives, but in order to contribute to economic growth, their essence must be embodied within an active factor of production. In fact, each of these so-called "disembodied" forms of technology all emerged from various forms of knowledge and technology embodied in persons, machines, and nature. In the following section, we examine the role of endogenous technical change in our model.

9. Endogenizing technology deepening and growth

For every g_H, there are unique steady-state savings rates that fulfill three conditions: allocate the resources for the creation of g_H, induce depreciation, broadly defined, and finance the replacement investment necessary to enable the potential for technology deepening to fully materialize. Given these functions of savings, we are able to use our model to endogenize g_H, the rate of technical change and steady-state output growth. As such, we seek to derive and analyze the relationship $g_H = \beta s$, in which the rates of savings for physical capital, s_K, and human capital, s_L, drive Hicks-neutral technical change.

In Section 8, we derived the net steady-state savings rate for embodied physical capital, s_K*, as $(BK/Y)g_B$. Given a fixed value for BK/Y, we infer a fixed relationship between s_K and g_B, i.e., that portion of physical savings that impacts directly on the non-human physical capital stock. As such, we have implicitly derived an equation for g_B in terms of the savings rate and the average product of physical capital:

$$g_B = (Y/BK)s_K^*. \qquad (9a)$$

There exists a counterpart relationship for human capital technical change and the human capital net rate of savings:

$$g_A = (Y/AL)s_L^*. \qquad (9b)$$

We use Eqs. (9a) and (9b) to construct Fig. 3, which shows the linear relationship between rates of savings and technical change. Using $g_B = g_A = g_H = g_Y$, we construct Fig. 3 as shown for which we calibrate $\partial g_Y/\partial s^*$, the slopes of the savings functions, from the statistical estimates of the last section, i.e., those for g_H, s_K^*, and s_L^* for the U.S. economy.

As shown in Fig. 4, calibrated using approximations of U.S. data, rates of technical change and output growth are positive functions of both rates of savings and the average products of physical and human capital. As calibrated, the linearity of the relationships, implies that a 3% increase in the rate of physical savings coupled with a 6% increase in the rate of savings for human capital results in a one percent increase in the rates of growth of technical change and output. The interested reader may wish to reaffirm or recalibrate the relationship shown in Fig. 4 using data for other countries.

One potentially unsettling implication of the diagram shown in Fig. 4 is the strictly linear relationship between rates of savings, technical change, and economic growth. The linear relationship implies that a golden-rule-driven society might wish to dedicate far larger shares of its income to savings than presently observable. The linearity results from a core assumption of the model, i.e., constant returns to scale in both the short-run and the long-run, both between and within countries.

Our analysis assumes that as a fraction of total savings, the resources dedicated to innovation, i.e., R_i, i = K, L, represent fixed proportions, i.e., $R_K/S_K = v_K$ and $R_L/S_L = v_L$. That is, increases in s_K^* and s_L^* result in equi-proportional increases in g_B, g_A, and g_H. These relationships persist regardless of the level of development of the affected countries, i.e. their physical and human capital intensities and proximities to the international technology frontier. In fact, analyses and findings, such as those of Jones and Williams (2000, p. 65) and Coccia (2009, p. 433), strongly suggest diminishing returns to R&D activity. Regardless of the precise specification between the savings, innovation expenditures, and innovation outcomes, this model provides a heuristic context within which to nest the vast expanse of endogenous growth literature, including Aghion and Howitt (1992), Jones and Vollrath (2013) and a variety of the models reviewed by Sveikauskas (2007) and Hall (2009).

10. Interpretation and microeconomic foundations

Several cornerstones of this paper invite deeper analysis; in some cases clarification of the implied micro-foundations. These include the justification for the assumption of Hicks-neutral technical change, the distinction between disembodied and embodied technical change, and the microeconomic underpinnings of the Schumpeterian creative destruction process.

Independent of the requisite limitations on the nature of technical change in the Solow-Uzawa neoclassical growth model, a large literature has evolved addressing the likely bias of technical change. We do not presume to add to the formal analysis contributed by Kennedy (1964) and Samuelson (1965, Hicks-neutral), Acemoglu (1998, skill-biased), and others. Rather, we simply attempt to augment the argument on behalf of Hick-neutral technical change from the perspective of this paper.

A useful starting point is the observation of Hicks in his correspondence with Harrod (1999, p. 349):

> Your theory, as I now see it, is a long-period theory where (in equilibrium) the supply of capital (i.e. the stock of capital) is not an independent but a dependent variable, adjusting itself to the other data of the system, such as the rate of growth. In such a system a definition of "invention neutrality" such as mine is not possible; one has to have a definition of the same type as yours, into which the stock of capital does not explicitly enter. Basically this is because in your system it is the *same* equilibrium when capital has doubled and everything else has doubled, as it was before…." [20]

What is most telling in this paragraph is the perspective that Hicks and Harrod-neutral technical change are not competing substitutes; instead, Hicks offers the possibility that the two interpretations of technical change can be viewed as sequential and complementary. As such, Hicks' definition of technical change is analogous to our Stage I phase in which technical change is measured in relation to a given K-L ratio. From this perspective, Hicks' invention-neutral technical change is entirely disembodied; that is, it augments efficiency given the existing

[20] Hicks to Harrod, 30 January 1963, see also Hicks 1963, pp. 348-350, reproduced in Besomi, 1999.

physical inputs and factor intensities. By contrast, Hicks characterizes Harrod-neutrality as that in which capital – being an endogenous factor – has adjusted so that "capital has doubled and everything else has doubled." Hence, Hicks relates the Harrod-neutral portion of the technical change to our Stage II phase characterized by the investment process. Our innovation is to constrain the possibility of uneven outcomes of Hicks-neutral technical change in the Solow model, so that all of the factor-augmenting inputs uniformly double.

What requires technical advance to be roughly balanced as between humankind and the physical world? From the Pyramids, to the steam engine, to computing and space travel, the processes of transformation of the physical environment and the human brain have been inseparable. Everything in our material world that results from innovation, conventionally understood as emerging from human intelligence, transits through the human brain; likewise, all such innovation for the human body, including medical interventions and education, transit through the workshops of nature and the physical world, rendering the Stage I innovation workshops of nature and the human intellect inseparable.

In their paper "What Determines the Direction of Technological Progress?," Li and Bental (2016) conclude that "…in the long-run the direction (of technical change) depends only on the relative supply elasticities of primary factors with respect to their respective prices." This condition, according to Li and Bental, account for the fact that in the Solow model in which the physical labor supply is entirely inelastic while physical capital is infinitely elastic, with the exception of the special condition $\sigma = 1$, technical change is restricted to being purely labor augmenting. An implication of the Li-Bental model is that with the physical supplies of capital and labor, *both* being entirely inelastic with respect to their respective prices, such as required in our model in which physical supplies are restricted by nature's fixed endowment, capital and labor factor-augmenting technical change must be uniform and Hicks-neutral.

A second issue concerns the heterogeneous distribution of physical and human technology. A serious issue facing the model crafted above is its homogeneous nature. That is, in the Black Hole single-sector model, technical change sweeps uniformly over the entire domain of the physical world, rendering its entirety uniformly obsolete in some measure sufficient to warrant its replacement with new vintages of technology yielding a single uniform efficiency increase equal to g_H.

In fact, the structure of the physical and cognitive worlds exists along a highly heterogeneous distribution of knowledge and technology vintages. We imagine a production efficiency distribution, which drifts through the time dimension as new technologies appear at the right-hand side of the distribution replacing old technologies disappearing from the left-hand tail of the distribution.

By introducing the "replacement effect," in which competing entrants are incentivized to develop innovations that chase the profits of incumbent monopolists who are themselves inclined to dedicate fewer of their own resources to innovations that displace their already-established markets," Arrow (1962) likely laid the conceptual foundation for formally modeling Schumpeter's creative-destruction paradigm.[21] From the perspective of this model, whereas the successful competing entrant is poised to capture the full $[(1/1\text{-}\alpha)]g_H$ from the physical innovation in question, the incumbent suffers the loss of δ_H resulting from the technology displacing effect of the innovation.

In his Chapter 14, "Models of Schumpeterian Growth," Acemoglu (2009) deftly summarizes the large literature on Schumpeterian growth models, including those of Aghion and Howitt (1992) and Grossman and Helpman (1991). His "Baseline Model of Schumpeterian Growth" (p. 459) represents a synthetic model in which there is a continuum of machines used in the production of a unique final good. With a fixed number of machine varieties, the measure of machine efficiency can be normalized, so that the efficiency of each machine line can be denoted by $\nu \in [0,1]$. The driver of economic growth, g_H, leads to technical advance that enables the economy to ascend a "quality ladder" as new rungs emerge from the innovation process motivating the investment required to replace lower rungs. Within our model, each rung represents a fixed amount of matter, so that the replacement process involves new capital replacing old capital in equal quantities by moving the lower quality rungs to higher quality rungs of equal mass. Again, our model provides a comprehensive, heuristic setting for a range of models of creative destruction.

[21] Schumpeter, 1942.

11. Conclusion and Reflections

By enabling technical change in the steady state to be of the Hick-neutral variety, the model in this paper turns the conventional Solow-Uzawa model on its head. The size of the physical world remains fixed as measured in physical units; the effective supplies of capital and labor grow only through interlocking and equi-proportional advances in capital and labor-augmenting technology, thereby creating balanced growth. In order to sustain the steady-state, the annual rates of physical and human capital saving and investment exactly replace the depreciated physical and human capital stock, further creating and accommodating annual increments to technological advance, g_H. The one-for-one process of creative destruction underscores the centrality of technology deepening as the core driver of balanced depreciation and efficiency-enhancing replacement investment. The model developed in this paper underscores the importance of knowledge creation; the challenge is how to understand and coordinate the interaction of nature's innovation workshop and the humankind's innovation workshop in service to humankind's objective functions and optimizing behavior, given the constraints of nature's fixed endowment.

By establishing savings rates as the exogenous policy vehicle for framing the innovation that drives destructive depreciation and creative investment, the model overturns the Solow model in which the savings rate is also exogenous but entirely passive as a driver of steady-state growth. Given the interdependence of savings and investment, the model sets a context for nesting the rich field of endogenous growth theory.

The model bushes up against several key public policy implications. The central public policy issue that emerges from the model is the determination of the optimal rates of savings and innovation investment. While certain literature, including Nuno (2010) in a general equilibrium context, Jones and Williams (2000) within an endogenous growth model, and Scotchmer (2004) within the context of optimal patent design, explores this issue, the thrust of this model suggests that the issue of socially optimal levels of physical and human capital savings should be far more extensively addressed and debated than evidence by the existing literature.

A second public policy issue arising from this model is the intrinsic link between creation, associated with innovation, and destruction or depreciation resulting from the same innovation. By centering the model on the creation-destruction nexus, the model engages with the public discourse on the consequences of trade and technical change for winners and losers with implications for public policy. While the model underscores the critical role of savings, it also underscores the negative externality that arise from the essential destruction and displacement that results from technology deepening. The inextricable link between creation and destruction invites deeper attention to the role of institutions and institutional economics in enabling the central economic condition of technology-deepening to achieve real-world Pareto-improving outcomes.

A final issue, one that bedevils the Solow model, is an issue that this model does not readily address; that is, an explanation of the dramatic difference among countries in living standards. The model does contain elements that imply catch-up. By inferring the multiplicative steady-state savings equations – $s_K = g_H(BK/Y)$ and $s_L = g_H(AL/Y)$ – the model suggests that for given rates of physical and human capital savings and investment, countries with low capital intensities – and high marginal productivities – should be able to achieve higher levels of technological advance. In this sense, by generating higher intensities of physical and human capital, the process of development results in a decline in the Beta-coefficients that mediate between savings, technology deepening, and growth.[22] That is, growth includes the seeds of slowdown.

And yet, by bundling the rates of technical change, depreciation, investment, and savings rates together, this model further diminishes the role of savings as the candidate explanation of cross-country income differences. Having demonstrated that differences in savings rates fall far short of explaining income differences among rich and poor countries, Romer (1994) bounds the role of saving as the principal factor determining persistent differences in long-run rates of growth. In our model, the other candidate explanation of persistent income differences is either physical resource endowments, K and/or L, and/or differences in original technology and institutional endowments, embedded in B and/or A, which together also enter into the Beta-factor that intermediates between savings and innovation.[23] The recognition that original

[22] The relationships $BK/Y = s_K^*/g_Y$ and $AL/Y = s_L^*/g_Y$ also imply Piketty's "second fundamental law of capitalism" (2015), i.e., if rates of output growth fall over time and savings rates (for physical capital) remain fixed, capital intensity – and unequal income distribution – increase. Piketty may also wish to extend his "fundamental law" to human capital.

[23] This literature, for example, includes Barro and Sali-i-Martin (1995) who identify heterogeneous policies toward intellectual property as a source of divergent rates of cross-country economic performance.

resource endowments matter have led to a variety of methodological innovations in the applied development and growth literature.[24]

Solow appears to have been inspired to formulate his neoclassical growth model in response to the Harrod-Domar condition that Solow characterized as follows (1956, p. 65):

> The characteristic and powerful conclusion of the Harrod-Domar line of thought is that even for the long run the economic system is at best balanced on a knife-edge of equilibrium growth. Were the magnitudes of the key parameters the savings ratio, the capital-output ratio, the rate of increase of the labor force - to slip ever so slightly from dead center, the consequence would be either growing unemployment or prolonged inflation.

Ironically, in formulating his alternative neoclassical steady state to resolve the Harrod-Domar knife-edge problem, as confirmed by Uzawa (1961), Solow (1956) created an equally vexing knife-edge condition – a description of long-run growth that precludes the possibility of capital-augmenting technical change once the parameter σ "slips ever so slightly" from unity. This paper may serve to illustrate how the knife-edge problems that emerge in economic literature are generally artifacts of the models that embody them – susceptibilities resulting from the quest for parsimony, not intrinsic features of the physical or economic world that we model and analyze. Quite possibly, this research contribution also conveys vexing limits and puzzles to our understanding of growth, such as those embedded in the Harrrod-Domar and Solow models. These present as welcome challenges for further research.

[24] See, for example, Glaeser et al, 2004.

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Figure I. Solow with Labor-Augmenting Technical Change, σ =1

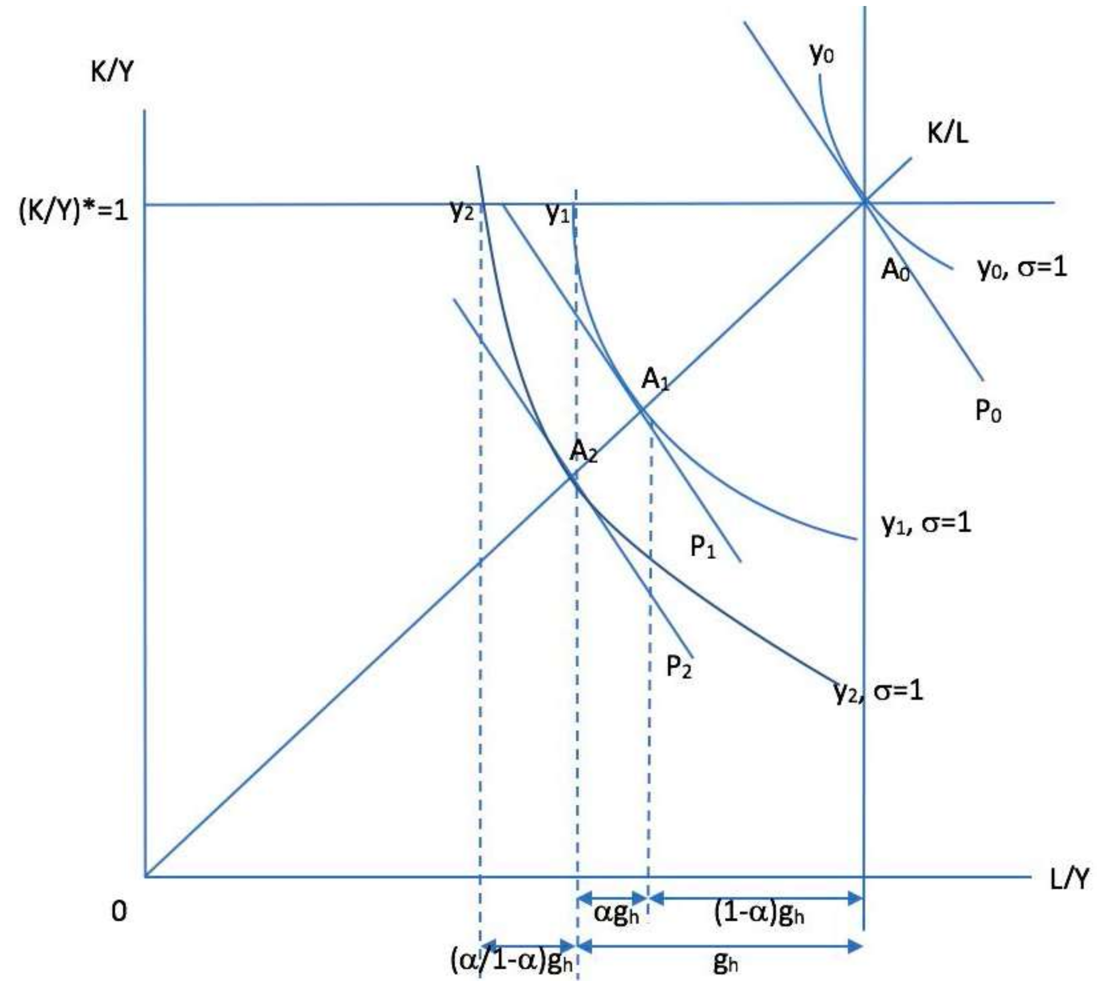


Figure 2. Solow with Hick's Neutral Technical Change, σ = 1 and σ < 1

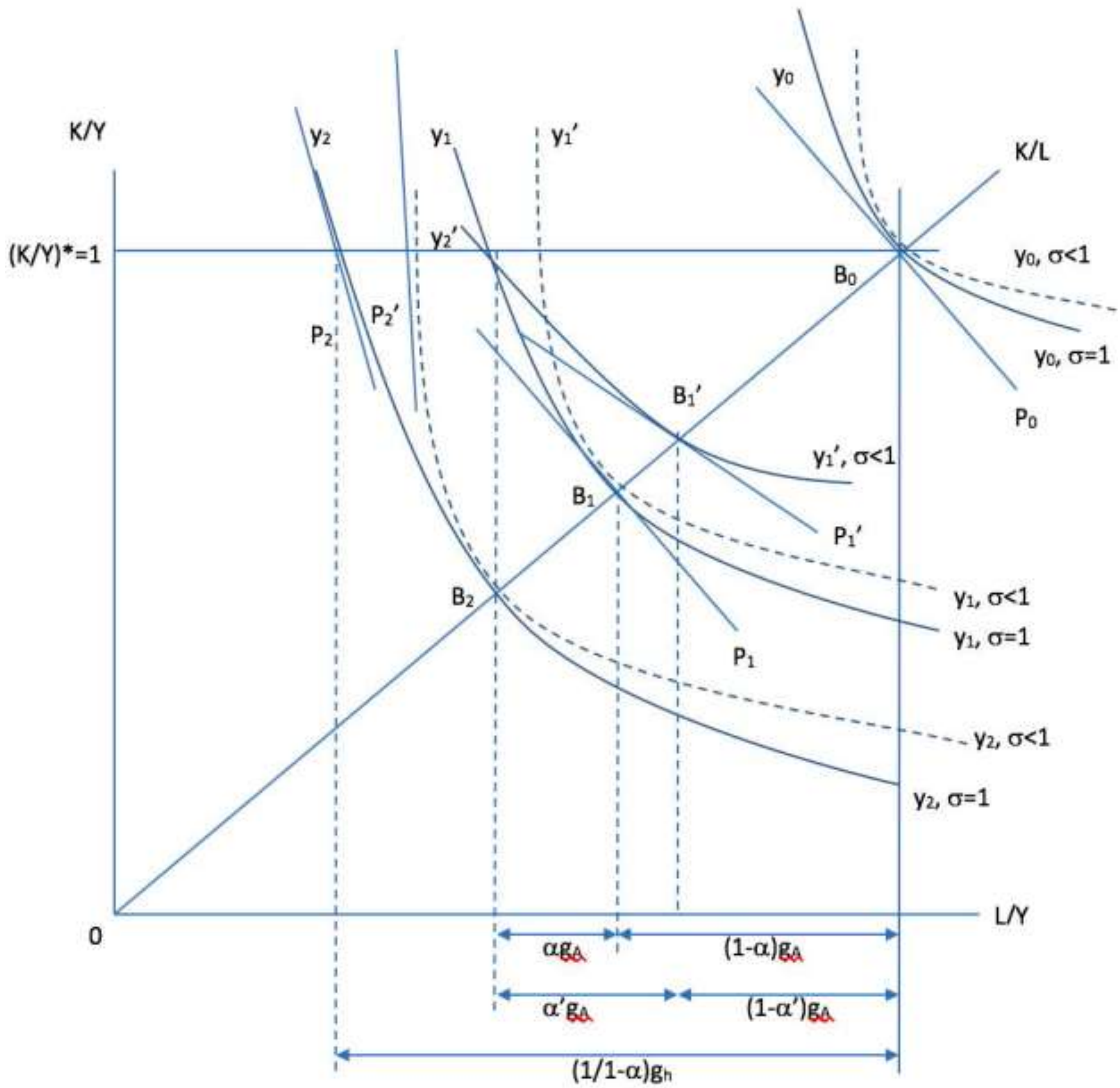


Figure 3. Hicks-neutral technical change with technology deepening only

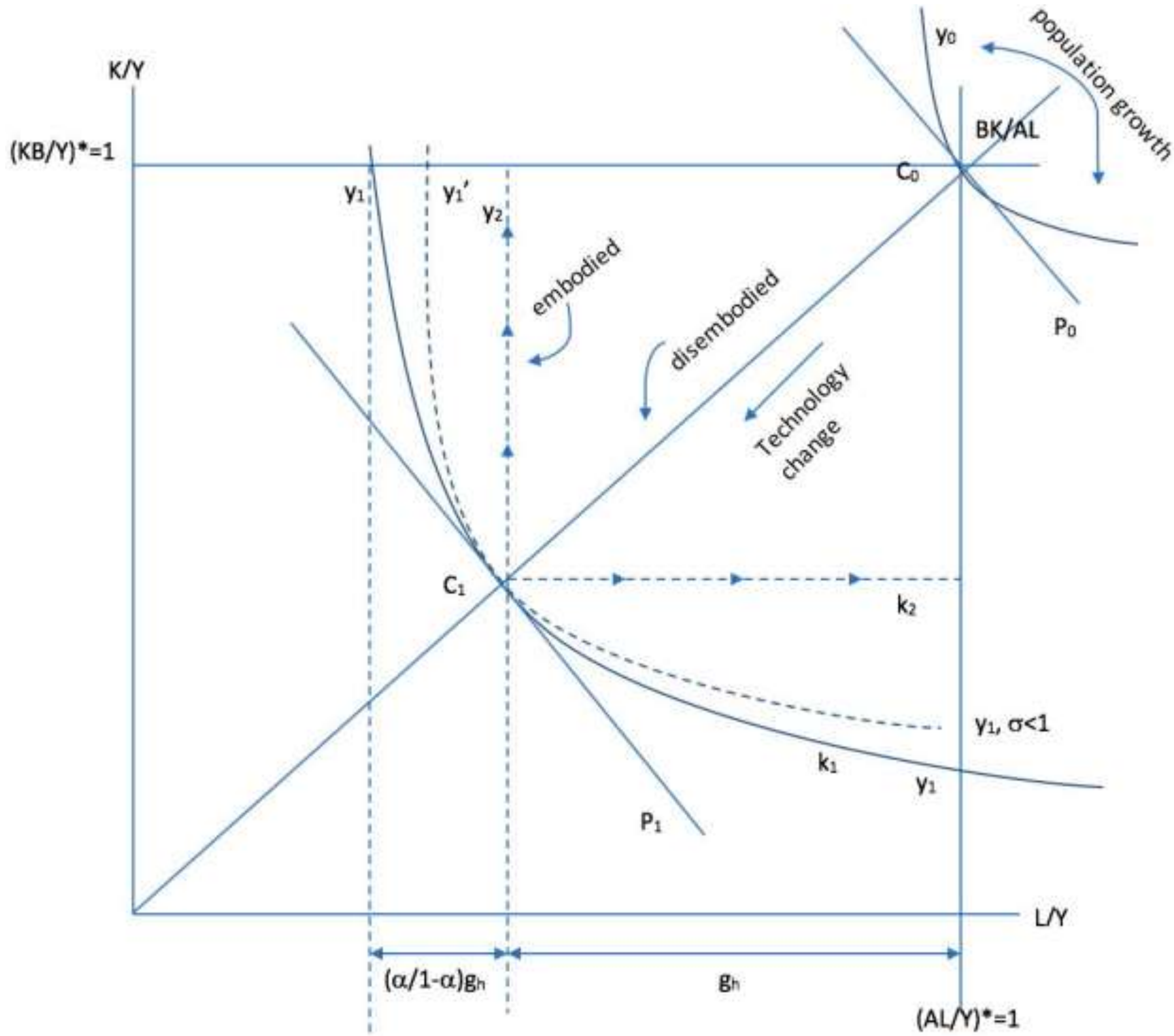
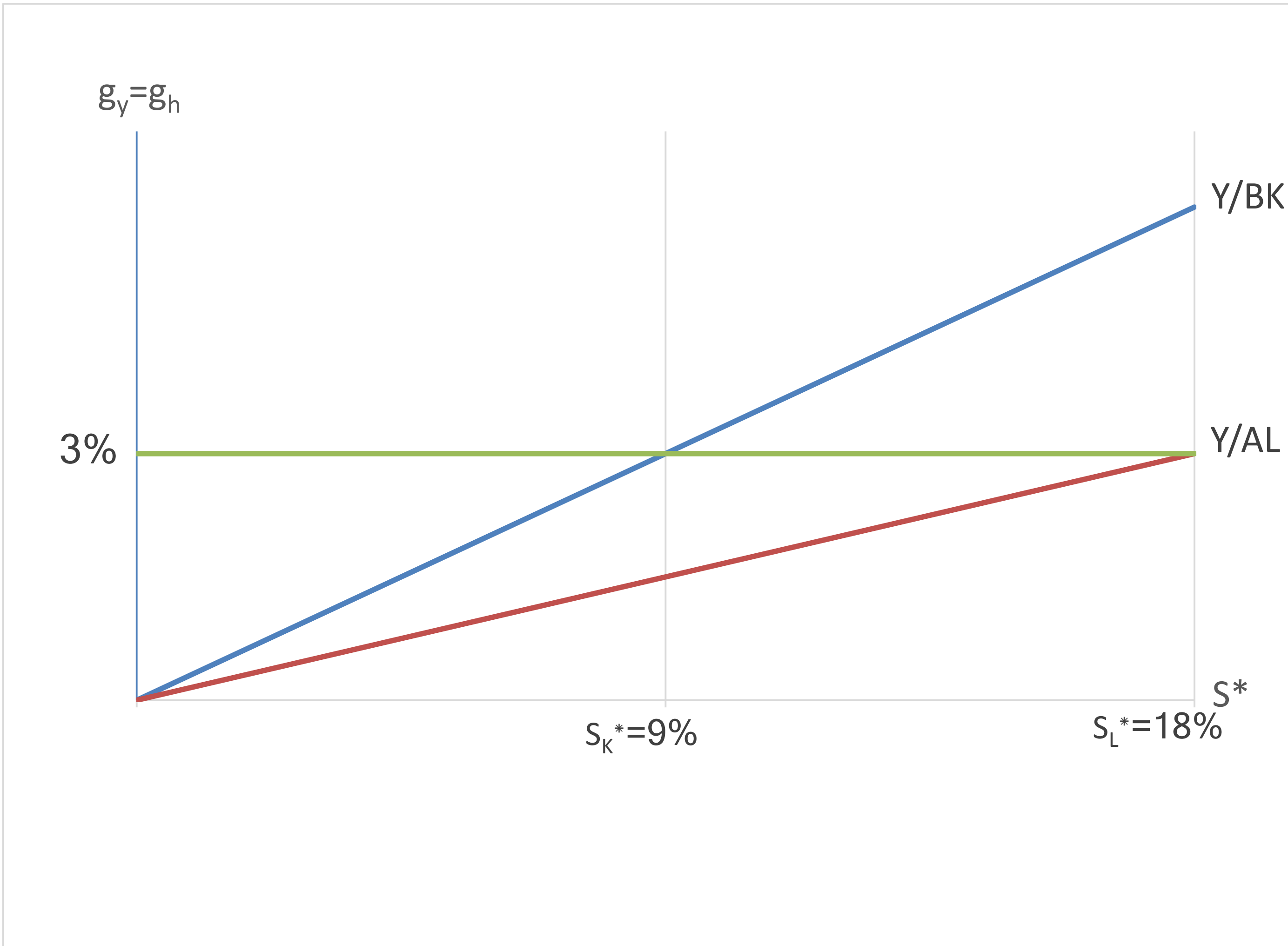


Figure 4. Savings, technical change, and growth



Annex I. Glossary

Term

g_B = capital augmenting technical change
g_A = labor (human) capital-augmenting technical change
g_H = Harrod-neutral technical change

Measures of depreciation:
$\delta_K = [\alpha/(1-\alpha)]g_B$ (physical capital)
$\delta_L = [(1-\alpha)/\alpha]g_A$ (human capital)

Measures of physical investment:
$g_K = [1/(1-\alpha)]g_B$ (gross)
$g_K^* = g_B$ (net)
$g_K' = [\alpha/(1-\alpha)]g_B$ (physial replacement)
$g_K'' = g_K' - \delta_K = 0$ (net physical replacement)

Measures of physical savings:
$s_K = [1/(1-\alpha)]g_B(BK/Y)$ (gross)
$s_K^* = g_B(BK/Y)$ (net)
$s_K' = [\alpha/(1-\alpha)]g_B(BK/Y)$ (replacement)
$g_K' - \delta_K = 0$ (net replacement)
$s_K = s_K^* + s_K'$

Measures of human capital investment:
$g_L = (1/\alpha)g_A$ (gross)
$g_L^*= g_A$ (net)
$g_L' = [(1-\alpha)/\alpha]g_A$ (physical replacement)
$g_L'' = g_L' - \delta_L = 0$ (net physical replacement)

Measures of human capital savings:
$s_L = (1/\alpha)g_A(AL/Y)$ (gross)
$s_L^* = g_A(AL/Y)$ (net)
$s_L' = [(1-\alpha)/\alpha]\ g_A(AL/Y)$ (replacement)
$s_L'' = s_L' - \delta_L = 0$ (net replacement)
$s_L = s_L^* + s_L'$

Annex II. The Model
(may be interpreted as amendments to the Solow model)

$Y = [\pi(BK)^{(\sigma-1)/\sigma} + (1-\pi)(AL)^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)}$ (production technology)

Assumption, K and L are fixed by nature's endowment; but highly malleable with applications of B and A, respectively.

$B/KY = v$; $AL/Y = u$, $g_B = g_A = g_H$ (fixed steady state technology-output ratios and Hicks-neutral technical change)

$gy = \alpha(g_B + g_K) + (1-\alpha)(g_A + g_L)$ (Solow steady state version; g_K, g_L perfectly supply elastic such that the technology muliplіers are $[1/(1-\alpha)]gB$ and $(1/\alpha)gA$, respectively; this model, g_K, $g_L = 0$, i.e., perfectly inelastic, constraining both technology multipliers to unity)

Equations of motion in the steady state with embodied technical change:

$d(BK)/BK = g_B + g_K' = s_K Y/BK - \delta_K$
$g_K' - \delta_K = g_K' - [\alpha/(1-\alpha)]g_B = 0$ (fixed physical capital assumption)
$d(AL)/AL = g_A + g_L' = s_L Y/AL - \delta_L$
$g_L' - \delta_L = g_L' - [(1-\alpha)/\alpha]g_A = 0$ (fixed physical labor assumption)

$s_K^* = g_B(BK/Y)$ (physical capital net savings rate)
$s_L^* = g_A(AL/Y)$ (human capital net savings rate)

$g_B = g_H = (Y/BK)s_K^*$ (endogenous growth equation – physical capital technical change)
$g_A = g_H = (Y/AL)s_L^*$ (endogenous growth equation – human capital technical change)

Factor shares:

$\alpha/(1-\alpha) = [\pi/(1-\pi)][(BK/AL)]^{(\sigma-1)/\sigma}$

$$g_L = \int_{t-65}^{t-20} (n_t/45)$$

Measurable steady-state requirements (for this model):

$\boldsymbol{\delta_K = [\alpha/(1-\alpha)]g_Y}$
$\boldsymbol{\delta_L = [(1-\alpha)/\alpha]g_Y}$

$\mathbf{s_K^*/g_Y = (BK/Y)}$
$\mathbf{s_L^*/g_Y = (AL/Y)}$